NON LINEAR BUCKLING OF COLUMNS

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Abstract:

The geometric non-linear total potential energy equation is developed and extended to study the behavior of buckling and deflection beyond the bifurcation point and showing columns resistance beyond the Euler load.

Three types of boundary conditions are studied (pin ended, fixed ended and cantilever). The equation of non-linear total potential energy is solved by exact method (closed form solution) and compared with other approximated methods (Rayleigh- Ritz, Koiter's theory and non-linear finite difference method). The agreement is found quite enough and satisfactory for most situations of practical cases.

Key words:

Bifurcation, Buckling, Columns, Finite difference method, Koiter's theory, nonlinear buckling and Rayleigh- Ritz method.

.(Euler load) (bifurcation point)
Rayleigh- Ritz,)
(Koiter's theory and non-linear finite difference method

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Notations:

E: Modulus of Elasticity.
I: Moment of inertia.
L: Length of the member.
P: Longitudinal axial load.
P_E: Euler load.
s: Non-linear longitudinal coordinate.
u: Longitudinal displacement.
w: Transverse deflection.
w_{max}: Maximum deflection of the member.
x: Longitudinal coordinate.
$\prod$: Total potential energy.
θ : Deflection angle.
Φ : Curvature.

Introduction:

Stability of a body is that condition [1-7], if after some slight disturbance in the configuration the body returns to the original configuration, this condition is satisfied when there is no change in the total potential energy as the position is varied. In other word, the stability is obtained when the total potential energy in minimum condition and un-stability occurred at maximum total potential energy. For conservative systems, the equilibrium configuration is corresponding to the minimum total potential energy.

Mathematically the stability problem is called an Eigen value problem, the critical load (buckling load) is an Eigen value load of the problem, and the deflection w(x) corresponding to the load is an Eigen function. Equilibrium method provides an infinite set of Eigen values (critical loads) where non-trivial configurations could satisfy the requirements of equilibrium and called modes of buckling. The lowest critical value is called the buckling load (Euler load).

The equation of minimum total potential energy (functional) or general governing differential equation of the problem can be solved by various techniques such as (closed form or exact solution, finite element method) or approximated methods such as (Rayleigh-Ritz method, Galerkin's method and finite difference method).

The bulk of buckling analysis that considered in most references is limited to the linearized Eigen value problems that define buckling load. In this study the geometrical non-linear total potential energy equation is developed for fixed ended and cantilever in addition to pin ended columns and solved by different methods.

1. Exact method (closed form solution).
2. An Intermediate method (Rayleigh- Ritz method).
3. Koiter's theory.
4. Non-linear finite difference method.

Derivation and solution:

I-Exact method:

The simplified functional for the total potential energy of columns can be written as following [1]:

$\Pi = EI/2 \int (\partial^2 w/ \partial x^2)^2 \, dx - P/2 \int (\partial w/\partial x)^2 \, dx$ --------------(1)

Where ($\partial^2 w/ \partial x^2$) represent the curvature (Φ) in linear case, but in case of geometrical non-linear bending, the exact non-linear curvature expressed as below as given in ref. [1]:

$\Phi = (\partial\theta/\partial s) = (\partial^2 w/ \partial x^2) / [1 + (\partial w/ \partial x)^2]^{3/2}$

Another form of curvature can be used in term of ($\partial w/ \partial s$) as below:

$\Phi = (\partial\theta/\partial s) = (\partial^2 w/ \partial s^2) / [1 - (\partial w/ \partial s)^2]^{1/2}$ -------------(2)

And the term [$1/2 \int (\partial w/\partial x)^2 \, dx$] represent the shortening of the member.

$\Delta L= [L - \int \cos(\theta) \, ds]$

The functional in equation (1) for the geometrical non-linear case becomes:

$\Pi = EI/2 \int (\partial\theta/\partial s)^2 \, ds - P [L - \int \cos(\theta) \, ds]$ --------------(3)

Minimizing (Π) with respect to (θ) and setting to zero ($\partial\Pi/\partial s=0$) gives the following Euler – Lagrange equation.

$EI \; (\partial^2 \theta/ \partial s^2) + P \sin(\theta) = 0$ --------------(4)

To simplify the above non-linear differential equation, multiply by ($\partial\theta/\partial s$) and rewrite in another form:

$d/ds \, [EI/2 \; (\partial\theta/ \partial s)^2 - P \cos(\theta)] = 0$ --------------(5)

Integrate the above equation directly to get:

$EI/2 \, (\partial\theta/ \partial s)^2 = P \cos(\theta) + C1$ --------------(6)

For pin ended columns the boundary condition at the ends (x=0 and L) are ($\partial\theta/ \partial s= 0$), this means that ($\theta$) is constant at the ends and vary between ($\pm \alpha$); where (α) is the angle at the ends.

For fixed ended columns, the variation of (θ) between ($\pm \alpha$) occurred at (x=L/4 and x=3/4 L).

By applying the boundary conditions; $C1= - P \cos(\alpha)$

Hence:

$EI/2 \, (\partial\theta/ \partial s)^2 = P [\cos(\theta) - \cos(\alpha)]$

or $\partial\theta/ \partial s = \sqrt{2P/EI \; [\cos(\theta) - \cos(\alpha)]}$ --------------(7)

Using $\cos(\theta)=1-2 \sin^2(\theta/2)$ and $\cos(\alpha)=1-2 \sin^2(\alpha/2)$

The above equation becomes:

$\partial\theta/ \partial s = \sqrt{4P/EI \; [\sin^2(\alpha/2) - \sin^2(\theta/2)]}$ --------------(8)

By separating variables, the following integration is obtained as following:

$L= \int ds = \sqrt{EI/4P} \int d\theta/ [\sin^2 \alpha/2 - \sin^2 \theta/2]^{1/2}$

Due to symmetry of he deflection curve about the line (x=L/2)

$L= \int ds = \sqrt{EI/4P} \int 2d\theta/ [\sin^2 \alpha/2 - \sin^2 \theta/2]^{1/2}$ -----------------(9)

The above equation to be changed to another form, using change of variables as following:

Let $\sin(\theta/2) = \sin(\alpha/2) \sin(\Phi)$ and $\rho= \sin(\alpha/2)$

With some algebric manipulation, the above equation can be written in the following manner:

$L = \sqrt{EI/P} \int 2d\Phi / [\,1 - \rho^2 \sin^2 \Phi\,]^{1/2}$ -----------------(10)

Where at ends for pin end column and at (x=L/4, x=3/4L) for fixed ended column $\theta = \pm\alpha$; then $\Phi = \pm \pi/2$

Introducing of the linear Euler load ($P_E = \pi^2 EI/L^2$) for pin ended column and ($P_E = 4\pi^2 EI/L^2$) for fixed ended column, [1, 2 & 6] , the above equation becomes:

$\sqrt{P/P_E} = 2/\pi \int d\Phi / [\,1 - \rho^2 \sin^2 \Phi\,]^{1/2}$ -----------------(11)

the integration limits vary from (0) to (L) for pin ended column while the limits vary from (L/4) to (3/4 L) for fixed ended column.

Where P/P_E is the ratio of nonlinear to linear buckling load (Euler load).

The maximum buckling deflection (w_{max}) of the column is related with the ratio (P/P_E), for this purpose, use $\sin(\theta) = \partial w/\partial s$, accordingly, the Euler-Lagrange equation (4) can be written as follows:

$EI\, \partial^2\theta / \partial s^2 + P\, \partial w/\partial s = 0$ ------------(12)

Integrate both sides and rearrange to get the following equation.

$w = -\,EI/P\; d\theta/ds + C2$

at the ends, $w = 0$ and $\partial\theta/\partial s = 0$

Thus C2=0

Use equation (7) to replace ($\partial\theta/\partial s$), to get:

$w = -\,\sqrt{2EI/P}\,[\,\cos(\theta) - \cos(\alpha)\,]^{1/2}$

At mid span, $\theta = 0$ and $w = w_{max}$

$w_{max} = -\,\sqrt{2EI/P}\,[\,1 - \cos(\alpha)\,]^{1/2}$ -------------(13)

Introducing Euler load (P_E) and using some trigonometric identity, the above equation can be written in another form:

$w_{max}/L = 2/\pi\,[\sin(\alpha/2) / \sqrt{P/P_E}\,] = 2/\pi\,(\,\rho / \sqrt{P/P_E}\,)$ -------------(14)

The equation is same for pin ended and fixed ended column

For cantilever column, the boundary conditions are different:

At x= 0, $\theta = 0$ and at x= L, $\theta = \alpha$; $\partial\theta/\partial s = 0$ and Euler load = $\pi^2 EI/4L^2$

Applying the above boundary conditions and introducing Euler load and using the same previous procedures, the final equation of the deflection becomes:

$w_{max}/L = 4/\pi\,[\,\sin(\alpha/2) / \sqrt{P/P_E}\,] = 4/\pi\,(\,\rho / \sqrt{P/P_E}\,\}$ -------------(15)

Procedures of solution: -

1- For known value of (P/P_E), the value of [$\rho = \sin(\alpha/2)$] is determined from equation (11), by trail and error method to satisfy the value of the integration to be equal to root of the ratio (P/P_E), using numeric integration method.
2- Use equation (14) or equation (15) to determine the value of (w_{max}/L) for the known values of (P/P_E) and (ρ) which determined from the previous step.
3- Repeat the procedures for other values of (P/P_E).
4- Tabulate the results and plot the relationship of (P/P_E) versus (w_{max}/L).

II-An Intermediate theory (Rayleigh-Ritz method): -

The total non-linear potential energy equation (3) and the non-linear curvature given in equation (2) are expressed in binomial expansion form using power series method as shown below [1]: -

$$\Phi \approx \partial\theta/\partial s \approx (\partial^2 w/\partial s^2)\,[1 + ½\,(\partial w/\partial s)^2] \qquad \text{-------------(16)}$$

The term representing the shortening of the member ($L - \int \cos(\theta)\, ds$), also expressed in form of ($\partial w/\partial s$), the final form of equation (3) becomes:-

$$\Pi = EI/2\int(\partial^2 w/\partial s^2)^2\,[1+½(\partial w/\partial s)^2]^2 ds - P/2\int[1+ ¼\,(\partial w/\partial s)^2]\,(\partial w/\partial s)^2\, ds \qquad \text{---(17)}$$

In small deflection theory, the term containing $(\partial w/\partial s)^2$ is being small in comparison with unity and replace (s) by (x) the form of equation (17) return to the classical theory for linear case.

Applying Rayleigh-Ritz method and employing the Eigen functions that satisfy the boundary conditions of the column as in the following: -

1- Pin ended column:-

The Eigen function is taken as the following.

$$w(s) = A \sin(\pi s/L) \qquad \text{-------------(18)}$$

The assumed function is satisfies the boundary conditions at the ends

At s=0 and L; w=0

Substitute in equation (17) and integrate the functional to find the final form of (Π) as shown:

$$\Pi = EI\,\pi^2/4L\,\,[a^2 + ¼\, a^4 + a^6/32 - P/P_E\,(a^2 + 3/16\, a^4)\,] \qquad \text{-------------(19)}$$

Where $a = A\,\pi/L$

Extremizing of (Π) with respect of (a) to obtain:

$$a^4 + 16/3\,(1- ¾\, P/P_E)\, a^2 + 32/3\,(1- P/P_E) = 0 \qquad \text{------------(20)}$$

The above equation of 4th order is solved by Newton-Raphson method to determine the value of (a) corresponding to the ratio (P/P_E), noting that ($w_{max}/L = a / \pi$), then plot the relationship of (P/P_E) versus (w_{max}/L).

2- Fixed ended column:-

The suitable Eigen function which satisfy the boundary conditions of the two fixed ends can be taken as the following: -

$$w(s) = A\,[1- \cos(2\pi s/L)] \qquad \text{-------------(21)}$$

Following the same procedures as in pin- ended column, the final equation becomes:

$$a^4 + 4/3\,(1- ¾\, P/P_E)\, a^2 + 2/3\,(1- P/P_E) = 0 \qquad \text{------------(22)}$$

Where $a = A\,\pi/L$ and $w_{max}/L = 2\,a / \pi$

3- Cantilever column:-

The suitable Eigen function, which satisfies the boundary conditions, can be taken as the following.

$$w(s) = A\,[1- \cos(\pi s/2L)] \qquad \text{-------------(23)}$$

The final equation becomes:

$a^4 + 4/3\ (1- ¾\ P/P_E\ )\ a^2 + 8/3\ (1- P/P_E) = 0$ ------------(24)

Where $a = A\ \pi / 2L$ and $w_{max}/L = 2\ a / \pi$

III-Koiter's theory:-

Consider (u) to be the axial displacement of the centerline of the column and (w) to be the transverse displacement of this center line [1].

$dL = \sqrt{(1+\partial u/\partial x)^2 + (\partial w/\partial x)^2}\ dx$ ----------(25)

If the center line is assumed to be in-extensible, then dL=dx.

Thus:

$(1+\partial u/\partial x)^2 + (\partial w/\partial x)^2 = 1$ ----------(26)

and $\partial u/\partial x = \sqrt{(1- (\partial w/\partial x)^2} -1$ ----------(27)

let $\sin(\theta) = \partial w/\partial x$ and $\cos(\theta) = 1- \sin(\theta)$

Take the non-linear curvature

$\Phi = (\partial^2 w/\partial x^2\ ) / \cos(\theta) = (\partial^2 w/\partial x^2\ ) / \sqrt{[1 - \sin^2(\theta)]}$

Or $\Phi = (\partial^2 w/\partial x^2\ ) / \sqrt{[1-(\partial w/\partial x)^2]}$ -------------(28)

This is exactly same as equation (2)

Using the above equations, the total potential energy equation can be written in the following form: -

$\Pi = EI/2 \int (\partial^2 w/\partial x^2\ )^2 / [1-(\partial w/\partial x)^2] dx - P \int [\sqrt{(1- (\partial w/\partial x)^2} -1]\ dx$ ------(29)

Now use the dimensionless variables ($\zeta = x/L$, $\psi = w/L$, $\lambda = P/P_E$)

Using power series expression and some algebric steps, the total potential energy can be expressed as the sum of a functional of order (2) and (4).

$P(\psi) = P_2(\lambda) + P_4(\lambda)$

$P_2(\lambda) = \int [(\psi'')^2 - \pi^2\lambda\ (\psi'\ )^2\ ]\ d\zeta$ --------(30)

$P_4(\lambda) = \int [(\psi'')^2\ (\psi')^2 - \pi^2\lambda/4\ (\psi')^4\ ]\ d\zeta$

Where (ψ) is a function of (ζ).

Equations (30) can be solved by substituting the suitable Eigen function that satisfy the boundary conditions of the column as in previous method.

1- Pin ended column:-

The Eigen function is taken as [$\psi = a \sin(\pi\ \zeta)$], the resulting equation is:

$\lambda = 1+ \pi^2 / 8\ \ a^2$ where $a = w_{max}/L$ and $\lambda = P/P_E$

The above equation can be written in the following form: -

$w_{max}/L = 2\ \sqrt{2} / \pi\ \ (P/P_E -1)^{1/2}$ -----------(31)

2- Fixed ended column:-

The Eigen function is taken as $\psi = a\ [1- \cos(2\pi\ \zeta)]$, the resulting equation is:

$\lambda = [1+ ½\ \pi^2\ a^2\ ] / [1+ 3/8\ \pi^2\ a^2\ ]$ where $a = ½\ (w_{max}/L)$

$w_{max}/L = \sqrt{2} / \pi\ \ [(P/P_E -1) / (1- ¾\ P/P_E)]^{1/2}$ ----------(32)

3- Cantilever ended column:-

The Eigen function is taken as $\psi = a\,[1-\cos(\pi\,\zeta/2)]$, the resulting equation is:
$\lambda = [1+\pi^2/16\,a^2]$ where $a = w_{max}/L$
$w_{max}/L = 4/\pi\,(P/P_E - 1)^{1/2}$ -----------(33)

IV-Non-linear finite difference method.

The non-linear equation of bending of a member subjected to axial force can be written as the following:
$EI\,(\partial^2 w/\partial x^2)\,/\,\sqrt{[1-(\partial w/\partial x)^2]} = -P\,w$
Re-arrange the above equation to obtain:
$(\partial^2 w/\partial x^2) + P/EI\,\sqrt{[1-(\partial w/\partial x)^2]}\,w = 0$ -------------(34)
The equation expressed in finite difference form using central difference method.
$(w_{i+1} - 2\,w_i + w_{i-1})/h^2 + P/EI\,\sqrt{[1-(w_{i+1}-w_{i-1})/2h)^2]}\,w_i = 0$
Or in simpler form: -
$w_{i+1} + A_i\,w_i + w_{i-1} = 0$ ------------(35)
Where $A_i = K\,[\sqrt{[1-(w_{i+1}-w_{i-1})/2h)^2}] - 2$ and $K = P\,h^2/EI$
$h = L/n$, where (n) is the number of segments.

Procedures of solution:

1-Assume a specified value of (w_{max}/L).
2-Assume suitable Eigen function which satisfy the boundary condition of the column (pin ended, fixed ended and cantilever) as in previous methods.

$w = b\,\sin(\pi x/L)$ for pin ended column.
$w = b\,[1-\cos(2\pi x/L)$ for fixed ended column.
$w = b\,[1-\cos(\pi x/2L)$ for cantilever column.

3-Use the assumed displacement function to find the initial values of (w_i) at all nodes to start the solution.
4-Substitute the known values of (w_i) in equation(35) to form a set of equations

$[A]\,\{w\} = 0$
Where [A] is a coefficient matrix
This equation is satisfied in two conditions

i- Either $\{w\} = 0$; this gives trivial solution, or
ii- Determinate of $[A] = 0$; this gives the Eigen value of the problem (K)

5-Find $P = EI/h^2\,K$
6-Find the ratio (P/P_E) corresponding to the assumed value of (w_{max}/L).
7-Repeat the above steps for other values of (w_{max}/L).
8-Plot the relation of (w_{max}/L) versus (P/P_E).
9-For fixed ended and cantilever columns, apply the same procedures using the suitable Eigen function and same changes in the formulation of the non-linear finite difference equation corresponding to the boundary conditions of the problem.

Discussions and conclusions:

The analysis of buckling that considered in most references is limited to the linearized Eigen value problems that define the buckling load. This research is a trail to study the geometric non-linear behavior of buckling beyond the buckling load for different types of columns (pin ended, fixed ended and cantilever). Derivations and results of exact theory are compared with different methods (Rayleigh-Ritz method, Koiter's theory and non-linear finite difference method).

Fig(1) shows the comparison of exact solution for all column types and show that the curves are tangent to the horizontal axis at the load ratio (P/P_E=1), and then the value of the deflection increased with increasing of (P/P_E). This point is called as the bifurcation point. This behavior means that the column can withstand even higher loads beyond the bifurcation point (buckling load). This situation is called post-buckling stability and the deformation regime beyond this point is called post-buckling regime.

Figs(2,3 and 4) show the comparison of the result obtained from the different methods (Rayleigh-Ritz method, Koiter's theory and non-linear finite difference method) for pin ended, fixed ended and cantilever respectively. The results show that all solutions give the same behavior beyond the bifurcation point and agreement between the exact solution and these methods are quite good for ($w_{max}/L < 0.3$) for pin ended and fixed ended columns and ($w_{max}/L < 0.4$) for cantilever column, these limits are quite enough and more than satisfactory for most situations of practical cases.

The variation of results of all methods in comparison with the exact method is about (±4%) for pin ended columns and (±5%) for fixed ended columns for ($w_{max}/L < 0.3$) while (±4%) for cantilever columns for ($w_{max}/L < 0.4$).

Fig(1) shows that for constant value of (P/P_E), the cantilever column gives deflection value approximately twice than pin and fixed ended cloumns as shown below: at $P/P_E = 1.1$

Type	w_{max}/L	Ratio
Pin ended column	0.26	1
Fixed ended column	0.255	1
Cantilever column	0.507	1.95

And for constant deflection ratio (w_{max}/L=0.4), the load ratio (P/P_E=1.5) for pin and fixed ended columns while (P/P_E=1.06) for cantilever column, this mean that cantilever column resist small extra load beyond the bifurcation point while pin and fixed ended resist much more up to (50%) beyond the bifurcation point.

at $w_{max}/L = 0.4$

Type	P/P_E
Pin ended column	1.5
Fixed ended column	1.5
Cantilever column	1.06

Increasing of the deflection (w_{max}/L) up to (0.25) cause increasing the load ratio (P/P_E) to (1.1), only (10%) increasing. But when (w_{max}/L) increase to (0.4) the non-linear load ratio (P/P_E) jumped to the value (1.5) this reflect and explain the effect of large deflection (geometric non-linearity) on the resistance of the columns. The response and behavior of pin and fixed ended columns beyond the bifurcation point are similar while the cantilever columns showed lesser effect.

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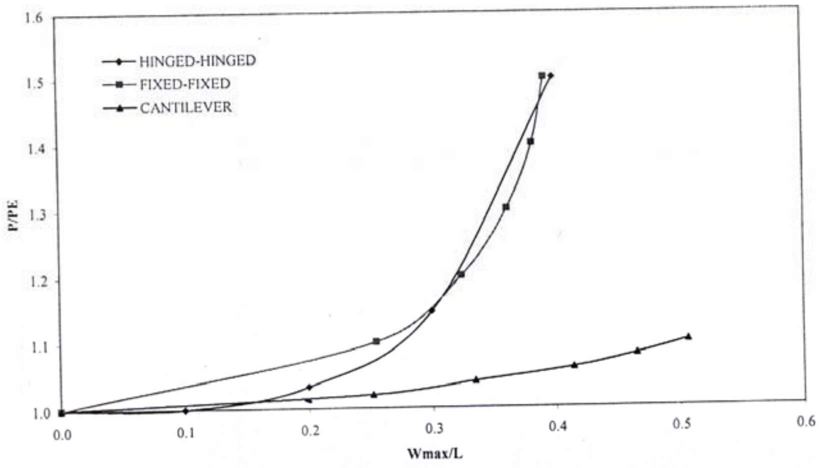

Fig (1): Comparison of (P/P_E) verses (w_{max}/L) for different columns.

EXACT METHOD		KOITER METHOD		RAY-RITZ METHOD		FINITE DIFF. METHOD	
P/PE	Wmax/L	P/PE	Wmax/L	P/PE	Wmax/L	P/PE	Wmax/L
1.000	0.0	1.000	0.0	1.000	0.0	1.0000	0.0
1.000	0.1	1.000	0.1	1.000	0.1	1.0100	0.1
1.033	0.2	1.033	0.2	1.033	0.2	1.0430	0.2
1.145	0.3	1.100	0.3	1.127	0.3	1.1080	0.3
1.500	0.4	1.190	0.4	1.260	0.4	1.2344	0.4
		1.300	0.5	1.435	0.5	1.7072	0.5

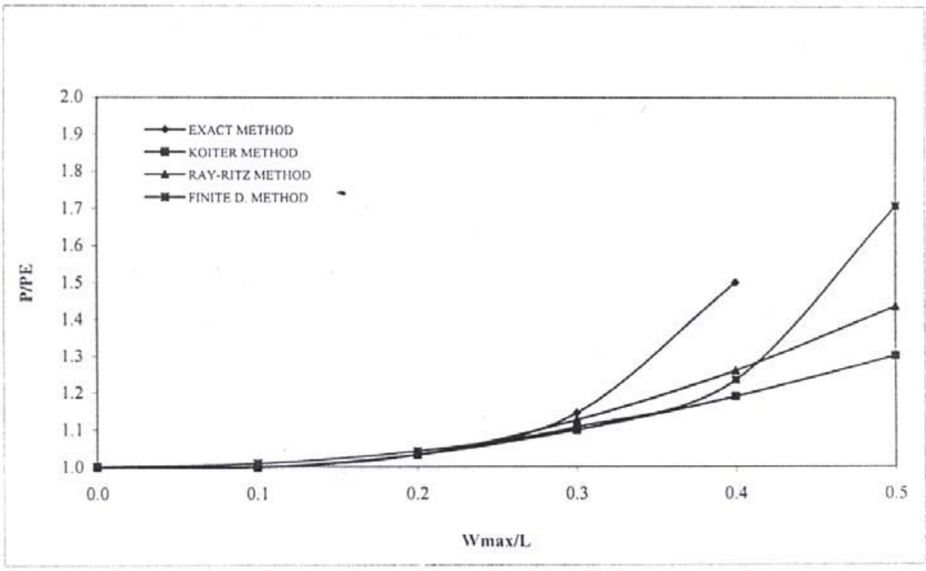

Fig (2): relationship of (P/P_E) verses (w_{max}/L) for pin ended column.

EXACT METHOD		KOITER METHOD		RAY-RITZ METHOD		FINITE DIFF. METHOD	
P/PE	Wmax/L	P/PE	Wmax/L	P/PE	Wmax/L	P/PE	Wmax/L
1.0	0.0000	1.0000	0.0	1.0	0.0000	1.0000	0.0
1.1	0.2550	1.0119	0.1	1.1	0.2600	1.0156	0.1
1.2	0.3243	1.0430	0.2	1.2	0.3500	1.0625	0.2
1.3	0.3615	1.0830	0.3	1.3	0.4180	1.1250	0.3
1.4	0.3825	1.1240	0.4	1.4	0.4725	1.2414	0.4
1.5	0.3930	1.1600	0.5	1.5	0.5200		0.5

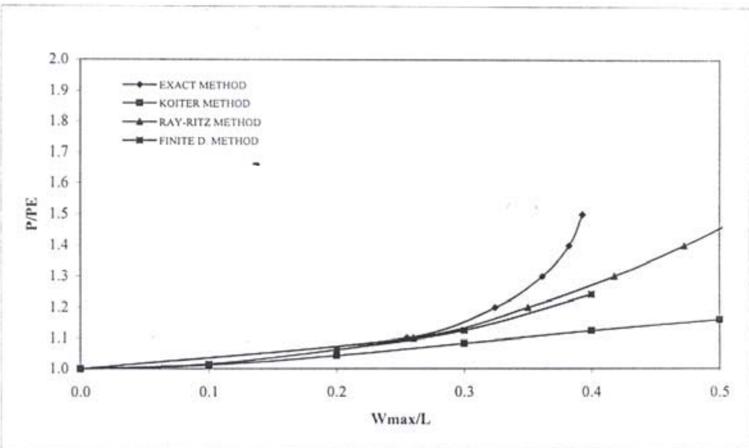

Fig (3): relationship of (P/P_E) verses (w_{max}/L) for fixed ended column.

EXACT METHOD		KOITER METHOD		RAY-RITZ METHOD		FINITE DIFF. METHOD	
P/PE	Wmax/L	P/PE	Wmax/L	P/PE	Wmax/L	P/PE	Wmax/L
1.00	0.0000	1.0000	0.0	1.0	0.0000	1.0000	0.0
1.02	0.2520	1.0062	0.1	1.1	0.4090	1.0100	0.1
1.04	0.3346	1.0250	0.2	1.2	0.5200	1.0227	0.2
1.06	0.4143	1.0555	0.3	1.3	0.5965	1.0450	0.3
1.08	0.4655	1.1000	0.4	1.4	0.6575	1.0955	0.4
1.10	0.5070	1.1540	0.5	1.5	0.7092	1.2100	0.5

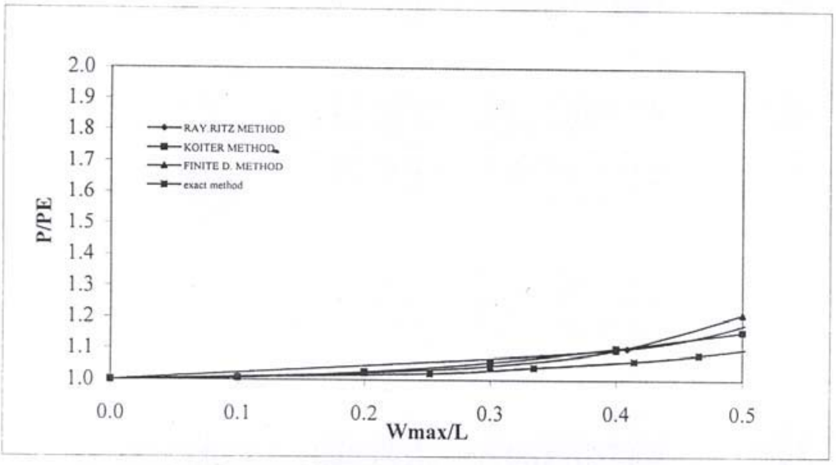

Fig (4): relationship of (P/P_E) verses (w_{max}/L) for cantilever column.